Derivatives of Multivariable Functions

Idea! The derivative measures change in output for corresponding small change in input... In some small direction

Def^n: Let f be a function of n-variables and $\vec{u}$ a unit vector in $\mathbb{R}^n$. Let $\vec{a} \in \text{dom}(f)$. The direction derivative of f at $\vec{a}$ in direction of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

Ex. Comp dir deriv of $f(x,y) = xy$ at $\vec{a} = \langle 1, 3 \rangle$ in direction $\vec{u} = \frac{1}{2}\langle \sqrt{3}, \sqrt{3} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} = \lim_{h\to 0^+} \frac{f(1+\frac{\sqrt{3}}{2}h, 3+\frac{\sqrt{3}}{2}h) - f(1,3)}{h}$

$\lim_{h\to 0^+} \frac{(1+\frac{\sqrt{3}}{2})(3+\frac{\sqrt{3}}{2}h) - 1\cdot 3}{h} = \lim_{h\to 0^+} \frac{(3 + h(\frac{3\sqrt{3}}{2} + \frac{1\sqrt{3}}{2}) + h^2) - 3}{h} = \lim_{h\to 0^+} \frac{h(2\sqrt{3} + h)}{h}$

$= \lim_{h\to 0^+} (2\sqrt{3}+h) = 2\sqrt{3} + 0 = 2\sqrt{3}$

Exercise: Find gen formula by substituting $\vec{a} = \langle x, y \rangle$

NB: The dir. deriv. is very general.
We want something like "rules" from Calc I...

Def^n: Let f be a function of n-variables and let $\vec{e}_k$ be k-th standard basis vector in $\mathbb{R}^n$, i.e. $e_k = \langle 0, 0, \ldots, \underset{k^{th}\ \text{position}}{1}, \ldots, 0 \rangle$

The k^{th} partial derivative of f (alt. the partial deriv of k^{th} pos. f with respect to x_k) is $D_{\vec{e}_k} f(\vec{a})$

Starting here

notation for part. deriv. $\to \frac{\partial f}{\partial x_k} = D_{\vec{e}_k} f$

$e_1 = \langle 1, 0 \rangle$

What's going on here?
Let's think about $n=2$: $f(x,y)$

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = D_{\vec{e}_1} f(\overset{x_1}{a}, \overset{x_2}{b})$

$= \lim_{h\to 0^+} \frac{f(\langle a,b \rangle + he_1) - f(a,b)}{h}$

$= \lim_{h\to 0^+} \frac{f(a+h, b) - f(a,b)}{h}$

Define $g(x) = f(x, b)$. This line becomes:

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h\to 0^+} \frac{g(a+h) - g(a)}{h}$ ← usual derivative all usual properties hold...

$= g'(a)$

by calc I

$\boxed{\begin{aligned} &\text{Ex. } f(x,y) = xy \\ &b = \pi \\ &g(x) = f(x, \pi) \\ &\quad = x\pi \end{aligned}}$

Point: $\frac{\partial f}{\partial x}$ is the "usual deriv." of f, pretending that every variable except for x is constant! (that was the point of g...)

Similarly $\frac{\partial f}{\partial y}$ is the deriv of f holding x constant...

curly ∂ represents multiple variables

Ex. Consider the partial derivatives of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\underbrace{xy + \sqrt{y} - \sin(x-y)}_{\text{use single variable deriv. properties}}] = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[\sqrt{y}] + \frac{\partial}{\partial x}[\sin(x-y)]$ chain rule

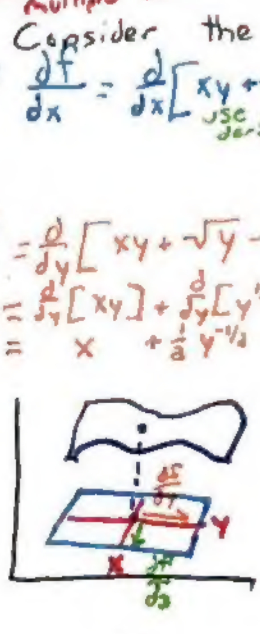
$= y\frac{\partial}{\partial x}[x] + \underset{\text{const. w.r.t. } x}{0} - \cos(x-y)\frac{\partial}{\partial x}[x-y]$

$= y(1) - \cos(x-y)(1) = y - \cos(x-y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[xy + \sqrt{y} - \sin(x-y)]$

$= \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial y}[y^{1/2}] - \frac{\partial}{\partial y}[\sin(x-y)]$

$= x + \frac{1}{2}y^{-1/2} - \cos(x-y)\frac{\partial}{\partial y}[x-y] = x + \frac{1}{2y^{1/2}} + \cos(x-y)$



Derivatives of Multivariable Functions cont.

Ex. Comp. partial deriv. of $f(x,y,z) = e^{x^2+y^2}\sin(xz)\cos(yz)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \cos(yz)\frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\right]$$

$$= \cos(yz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\sin(xz) + \frac{\partial}{\partial x}\left[\sin(xz)\right]e^{x^2+y^2}\right)$$

$$= \cos(yz)\left(2xe^{x^2+y^2}\sin(xz) + e^{x^2+y^2}z\cos xz\right)$$

$$= e^{x^2+y^2}\cos(yz)\left(2x\sin(xz) + z\cos(xz)\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \sin(xz)\frac{\partial}{\partial y}\left[e^{x^2+y^2}\cos(yz)\right]$$

$$= \sin(xz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\cos(yz) + \frac{\partial}{\partial x}\left[\cos(yz)\right]e^{x^2+y^2}\right)$$

$$= \sin(xz)\left(2ye^{x^2+y^2}\cos(yz) + e^{x^2+y^2}(-z\sin(yz))\right)$$

$$= \sin(xz)e^{x^2+y^2}\left(2y\cos(yz) - z\sin(yz)\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = e^{x^2+y^2}\left(\sin(xz)\cos(yz)\right)$$

$$= e^{x^2+y^2}\left(\frac{\partial}{\partial z}\left[\sin(xz)\right]\cos(yz) + \frac{\partial}{\partial z}\left[\cos(yz)\right]\sin(xz)\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) + (y(-\sin(yz))\sin(xz)\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) - y\sin(yz)\sin(xz)\right)$$

NB: higher order partial deriv. still make sense just like Calc I except now there's more

If $f(x,y)$ is given, the second order partials are:

$$\underbrace{\frac{\partial^2 f}{(\partial x)^2},\ \frac{\partial^2 f}{(\partial y)^2}}_{\text{"pure partial deriv"}},\ \frac{\partial^2 f}{\partial y\,\partial x},\ \frac{\partial^2 f}{\partial x\,\partial y}$$

"mixed partial deriv"

$\frac{\partial^2 f}{\partial x\,\partial y}$: deriv of y first then x

$\frac{\partial^2 f}{\partial y\,\partial x}$: deriv of x first then y

Ex. Comp. 2nd-order partial deriv. of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$

Earlier we computed:

$$\frac{\partial f}{\partial x} = y - \cos(x-y) \quad \& \quad \frac{\partial f}{\partial y} = x + \frac{1}{2}y^{-1/2} + \cos(x-y)$$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\left[y - \cos(x-y)\right] = \sin(x-y)$$

$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] = -\frac{1}{4}y^{-3/2} + \sin(x-y)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[y - \cos(x-y)\right] = 1 - \sin(x-y)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] = 1 - \sin(x-y)$$

Interlude: These are truly just calc I deriv...
Working w/ 1 variable at a time allows us to utilize calc I strategies!

Deriv of Multivariable Functions cont.

Backed to mixed partials (somehow using both variables)

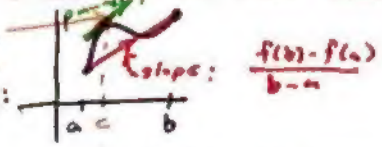
① Why were these equal in our example?
② Can we garentee this in future examples

Recall some Calc I, ~~Mean~~ Nice Average Value Thm...

Prop (Mean Value Thm): Let $f(t)$ be a function differentiable on (a,b) and cts on $[a,b]$.

There is a value $a<c<b$ such that $f'(c)(b-a) = f(b) - f(a)$

Idea: There is a pt, c, in (a,b) so that



Next time, we use MVT to prove the following:

Prop (Clairaut's Thm):

Suppose $f(x,y)$ has cont. second-order partial deriv. on some disk including pt. (a,b)

Then $\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}$